Tentamen Metrische Ruimten
18 april 2007, 09:00 - 12:00 uur, Examenhal

You can answer the exam in Dutch or English.

1. Consider $\mathbb{R}$ with the standard topology and the subset $H = [0, 1)$ with the induced subspace topology. For each one of the following subsets find its set of limit points, its closure and its boundary. Which of these sets are closed, which are complete and which are compact?
   (a) $(1/4, 3/4)$.
   (b) $(1/4, 1)$.
   (c) $[0, 1/2)$.
   (d) $[0, 1/2]$.
   (e) $\{1/n : n \in \{2, 3, 4, \dots\}\}$.
   (f) $\mathbb{Q} \cap (0, 1)$.
   $\mathbb{Q}$ is the set of rational numbers. All the properties (limit points, closure, boundary, closedness, compactness and completeness) should be examined with respect to the induced subspace topology in H. Support your answers by arguments.

2. (a) Give a direct proof that a sequentially compact metric space is bounded. 'Direct' here means 'do not use the fact that sequentially compact metric spaces are compact'.
   (b) Give an example of a metric space which is precompact (totally bounded) but not compact. $(0, 1)$

3. Suppose that $f : M \to M$ is a map of a complete metric space M, and that for some integer r, the iterated map $f^{(r)} = f \circ f \circ \cdots \circ f$ (r times) is a contraction. Prove that f has a unique fixed point $p \in M$.

4. Consider a sequence (f_n) of real-valued functions defined on $[0, 1]$, such that (f_n) converges uniformly to a function f, and that f_n is uniformly continuous on $[0, 1]$ for each n. Prove that f is uniformly continuous on $[0, 1]$.